Mixed-Integer Nonlinear Programming

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Abstract

Recently, the area of Mixed Integer Nonlinear Programming (MINLP) has experienced tremendous growth and a flourish of research activity. In this article we will give a brief overview of past developments in the MINLP arena and discuss some of the future work that can foster the development of MINLP in general and, in particular, robust solver technology for the practical solution of problems.

1 Introduction

Mixed Integer Nonlinear Programming (MINLP) refers to mathematical programming with continuous and discrete variables and nonlinearities in the objective function and constraints. The use of MINLP is a natural approach of formulating problems where it is necessary to simultaneously optimize the system structure (discrete) and parameters (continuous).

MINLPs have been used in various applications, including the process industry and the financial, engineering, management science and operations research sectors. It includes problems in process flow sheets, portfolio selection, batch processing in chemical engineering (consisting of mixing, reaction, and centrifuge separation), and optimal design of gas or water transmission networks. Other areas of interest include the automobile, aircraft, and VLSI manufacturing areas. An impressive collection of MINLP applications can be found in [14] and [15]. The needs in such diverse areas have motivated research and development in MINLP solver technology, particularly in algorithms for handling large-scale, highly combinatorial and highly nonlinear problems.

The general form of a MINLP is

$$\begin{array}{lll} \text{minimize} & f(x,y) & \\ \text{subject to} & g(x,y) \leq 0 & \\ & x \in X & \\ & y \in Y & \text{integer} \end{array} \tag{1}$$

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The function $f(x, y)$ is a nonlinear objective function and $g(x, y)$ a nonlinear constraint function. The variables x, y are the decision variables, where y is required to be integer[1] valued. X and Y are bounding-box-type restrictions on the variables. We refer to [9] for more information about MINLP fundamentals in textbook format.

2 Algorithms

MINLP problems are precisely so difficult to solve, because they combine all the difficulties of both of their subclasses: the combinatorial nature of mixed integer programs (MIP) and the difficulty in solving nonconvex (and even convex) nonlinear programs (NLP). Because subclasses MIP and NLP are among the class of theoretically difficult problems (*NP-complete*), so it is not surprising that solving MINLP can be a challenging and daring venture. Fortunately, the component structure of MIP and NLP within MINLP provides a collection of natural algorithmic approaches, exploiting the structure of each of the subcomponents.

Solution Approaches

Methods for solving MINLPs include innovative approaches and related techniques taken and extended from MIP. Outer Approximation (OA) methods [5, 6], Branch-and-Bound (B&B) [16, 23], Extended Cutting Plane methods [33], and Generalized Bender's Decomposition (GBD) [13] for solving MINLPs have been discussed in the literature since the early 1980's. These approaches generally rely on the successive solutions of closely related NLP problems. For example, B&B starts out forming a pure continuous NLP problem by dropping the integrality requirements of the discrete variables (often called the relaxed MINLP or RMINLP). Moreover, each node of the emerging B&B tree represents a solution of the RMINLP with adjusted bounds on the discrete variables.

In addition, OA and GBD require the successive solution of a related MIP problem. Both algorithms decompose the MINLP into an NLP subproblem that has the discrete variables fixed and a linear MIP master problem. The main difference between GBD and OA is in the definition of the MIP master problem. OA relies on tangential planes (or linearizations), effectively reducing each subproblem to a smaller feasible set, whereas the master MIP problem generated by GBD is given by a dual representation of the continuous space.

The approaches described above only guarantee global optimality under (generalized) convexity. Deterministic algorithms for global optimization of *nonconvex problems* require the solution of subproblems obtained via *convex relaxations* of the original problem in a branch-and-bound context, and have been quite successful in solving MINLPs [7, 29].

[1] Other special types of discrete variables known from the "linear world" such as SOS, semi-continuous, and semi-integer variables can also be handled by most algorithms.

3 Software

Although theoretical algorithmic ideas for solving MINLP have been around for a while, the practical implementation of such concepts is much more difficult. Memory limitations, efficient numerical linear algebra routines, suitable algorithmic tolerances, and determining default solver options are some of the key issues faced when extending algorithms to large-scale, general-purpose software. In this section we give a brief and possibly incomplete historical overview of practical general purpose MINLP software.

Commercial MINLP Software Packages

Best to our knowledge, the earliest commercial software package that could solve MINLP problems was SCICONIC [10, 27] in the mid 1970's. Rather than handling nonlinearities directly, linked SOS variables provided a mechanism to represent discretized nonlinear functions and allowed solving the problem via MIP. In the mid 1980's Grossman and Kocis [17] developed GAMS/DICOPT [12], a general purpose MINLP algorithm based on the outer approximation method. In the early 1990's LINDOs [25] and What's Best [24] B&B code using the Generalized Reduced Gradient (GRG) code for subproblems was extended to solve MINLPs.

Since then a number of excellent academic as well as commercial codes have surfaced, including alphaECP [34] and mittlp [28], both of which are based on extended cutting plane methods, and MINLP_BB [19] and SBB [12], which use branch-and-bound to solve relaxed NLP subproblems. Even on the frontier of global MINLP, reliable and large-scale packages have materialized including alphaBB [1] and BARON [29], which use convex relaxations in a branch-and-bound framework.

Modeling Languages

The emergence of *algebraic modeling languages* in the mid to late 1980's and early 1990's has greatly simplified the process of modeling, in particular the formulation of MINLP type problems. Also, from a MINLP solver perspective, a modeling system delivers reliable black-box-type function evaluations and first and second order derivative information. Finally, the common solver interface of a modeling system allows MINLP algorithms to deploy existing NLP and MIP solvers to solve subproblems in a seamless way. A collection of MINLP models can be found in libraries such as MacMINLP [18] (AMPL [11] models), chapter 12 of [8] (GAMS [4] models) and as a superset MINLPLib [4] (GAMS models). The latter is available as part of the *MINLP World*. MINLP World is a forum for discussion and dissemination of information about all aspects of MINLP [20].

4 Recent Developments

With the recent progress made in global optimization, the importance of modeling systems has taken on a more significant role. In particular, most global solvers require more than black-box function evaluations. These solvers need *structural information* of algebraic expressions to build convex relaxations. AlphaBB and the modeling language MINOPT [26], as well as the recent release of GAMS/BARON [29] have shown the feasibility of this concept.

Another important advancement is the implementation of *open algorithms*. AIMMS-OA [2] is an outer approximation method similar to GAMS/DICOPT, but with the distinct feature that it allows user modification for fine-tuning the method for a particular problem. Such an open approach allows advanced users to adjust the algorithm to suit the problem at hand.

Recent research has also focused on combining of Random Search (RS), such as Tabu, Scatter Search, Simulated Annealing or Genetic Algorithms, with NLP methods. Recent implementations like OQNLP [12, 30] and LaGO [21, 22] have proven to be quite successful.

Finally, the area of Disjunctive Programming uses disjunctions and logic propositions to represent the discrete decisions in the continuous and discrete space respectively. Disjunctive programs, conveniently modeled and automatically reformulated in big M or convex region models, give access to a rich area of applications. Widespread interest in such models has spawned a new computing environment (LogMIP [31]), developed specifically for generalized disjunctive programming.

5 Future Directions

Progress in the MINLP arena has been significant in recent years, and we are now able to solve large-scale problems efficiently using a wide variety of approaches. However, MINLP has yet to reach the level of maturity that MIP has achieved. While the MIP community has benefited greatly from *preprocessing* to reduce model sizes and to detect special structure, MINLP technology is still lagging behind. NLP and MINLP preprocessing, similar to global methods, will require the delivery of structural information from the modeling languages. Progress on reliable large-scale NLP codes with restarting capabilities will have an immediate impact on MINLP. Furthermore, combining individual algorithms (e.g. branch-and-bound and extended cutting plane method) with sophisticated search strategies (e.g. non-trivial B&B selection strategies) and heuristics to quickly determine integer solutions will help to close the gap. If research and development continues at the current level of activity, MINLP will soon achieve a stage of maturity enjoyed by the other areas in mathematical programming.

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